

CONSTRUCTING MODEL CATEGORIES WITH PRESCRIBED FIBRANT OBJECTS

ALEXANDRU E. STANCULESCU

ABSTRACT. We put a model category structure on the category of small categories enriched over a suitable monoidal simplicial model category. For this we use the model structure on small simplicial categories due to J. Bergner and a weak form of a recognition principle for model categories due to J.H. Smith. We give an application of this weak form of Smith's result to left Bousfield localizations of categories of monoids in a suitable monoidal model category.

There are nowadays several recognition principles which allow one to put a Quillen model category structure on a given category. One of them is the following (by now classical) theorem of J.H. Smith.

Theorem 0.1. [2, Theorem 1.7] *Let \mathcal{E} be a locally presentable category, W a full accessible subcategory of $\text{Mor}(\mathcal{E})$, and I a set of morphisms of \mathcal{E} . Suppose they satisfy:*

c0: W has the two out of three property.

c1: $\text{inj}(I) \subseteq W$.

c2: The class $\text{cof}(I) \cap W$ is closed under transfinite composition and under pushout.

Then setting weak equivalences: $=W$, cofibrations: $=\text{cof}(I)$ and fibrations: $=\text{inj}(\text{cof}(I) \cap W)$, one obtains a cofibrantly generated model structure on \mathcal{E} .

We can say that (a) in practice, it is condition c2 above that is often the most difficult to check, and (b) this result does not give any description of the fibrant objects of the resulting model structure. In this note we

(1) advertise an abstraction of a technique due to D.-C. Cisinski [5, Proof of Théorème 1.3.22] and A. Joyal (unpublished, but present in his proof, circa 1996, of the model structure for quasi-categories) which addresses both (a) and (b), in the sense that it makes c2 easier to check and it gives information on the fibrant objects, provided that other assumptions hold, and

(2) give an application of this technique to the homotopy theory of categories enriched over a suitable monoidal simplicial model category, and to left Bousfield localizations of categories of monoids in a suitable monoidal model category.

The paper is organized as follows. In section 1 we detail the above mentioned technique. In section 2 we prove our main result, Theorem 2.4, namely that the category of (small) categories enriched over a combinatorial monoidal simplicial model category having cofibrant unit and which satisfies the monoid axiom admits a certain model category structure. The proof uses the analogous model structure for categories enriched in simplicial sets, due to J. Bergner [3]. We modify one of the steps in Bergner's proof; this modification is a key point in our approach, and it enables us to apply the technique from section 1. We also fix (Remark 2.6), in an appropriate way, the mistake in [16]. In section 3 we extend a result of R. Fritsch and D.M. Latch [8, Proposition 5.2] to enriched categories; this is needed in the proof of the main result. The section is self contained. Motivated by considerations from [12], we apply in section 4 our technique from section 1 to study left Bousfield localizations of categories of monoids. Precisely, let LM be a left Bousfield localization of a monoidal model category M . We study the problem of putting a model category structure on the category of monoids M , somehow related to LM , when one does not know whether the monoid axiom holds in LM .

1. CONSTRUCTING MODEL CATEGORIES WITH PRESCRIBED FIBRANT OBJECTS

We recall from [6] the following definitions. Let \mathcal{E} be an arbitrary category and W a class of maps of \mathcal{E} . W is said to satisfy the **two out of six property** if for every three maps r, s, t of \mathcal{E} for which the two compositions sr and ts are defined and are in W , the four maps r, s, t and tsr are in W . W is said to satisfy the **weak invertibility property** if every map s of \mathcal{E} for which there exist maps r and t such that the compositions sr and ts exist and are in W , is itself in W . The two out of six property implies the two out of three property. The converse holds in the presence of the weak invertibility property.

The terminal object of a category, when it exists, is denoted by 1.

Lemma 1.1. (D.-C. Cisinski, A. Joyal) *Let \mathcal{E} be a locally presentable category, $(\mathcal{A}, \mathcal{B})$ a weak factorisation system on \mathcal{E} and W a class of maps of \mathcal{E} satisfying the two out of six property. Let \mathfrak{J} be a set of maps of \mathcal{E} . Let us call a map of \mathcal{E} which belongs to $\text{inj}(\mathfrak{J})$ a **naive fibration**, and an object X of \mathcal{E} **naively fibrant** if $X \rightarrow 1$ is a naive fibration.*

(a) *If $\text{cell}(\mathfrak{J}) \subset W$, then a map which has the left lifting property with respect to the naive fibrations between naively fibrant objects is in W .*

(b) *If, in addition to (a), a map between naively fibrant objects which is both a naive fibration and in W is in \mathcal{B} , then a map in \mathcal{A} is in W if and only if it has the left lifting property with respect to the naive fibrations between naively fibrant objects.*

(c) *If $\text{cell}(\mathfrak{J}) \subset \mathcal{A} \cap W$ and (b) holds, then a map $X \rightarrow 1$ is in $\text{inj}(\mathcal{A} \cap W)$ if and only if X is naively fibrant, and a map between naively fibrant objects is in $\text{inj}(\mathcal{A} \cap W)$ if and only if it is a naive fibration.*

Proof. (a) Let $i : A \rightarrow B$ be a map which has the left lifting property with respect to the naive fibrations between naively fibrant objects. Factor the map $B \rightarrow 1$ as $B \rightarrow \bar{B} \rightarrow 1$, where $B \rightarrow \bar{B}$ is in $\text{cell}(\mathfrak{J})$ and \bar{B} is naively fibrant. Then factor the composite map $A \rightarrow \bar{B}$ as $A \rightarrow \bar{A}$ in $\text{cell}(\mathfrak{J})$ followed by a naive fibration $\bar{A} \rightarrow \bar{B}$. The resulting commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & \bar{A} \\ i \downarrow & & \downarrow \\ B & \longrightarrow & \bar{B} \end{array}$$

has then a diagonal filler, and so the hypothesis and the two out of six property of W implies that i is in W .

(b) Let

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{v} & Y \end{array}$$

be a commutative diagram with i in $\mathcal{A} \cap W$ and p a naive fibration between naively fibrant objects. Factor v as a map in $\text{cell}(\mathfrak{J})$ followed by a naive fibration: $B \rightarrow \bar{B} \rightarrow Y$. Factor the canonical map $A \rightarrow \bar{B} \times_Y X$ as a map in $\text{cell}(\mathfrak{J})$ followed by a naive fibration: $A \rightarrow \bar{A} \rightarrow \bar{B} \times_Y X$. It suffices to show that the square

$$\begin{array}{ccc} A & \longrightarrow & \bar{A} \\ i \downarrow & & \downarrow \\ B & \longrightarrow & \bar{B} \end{array}$$

has a diagonal filler. The map $\bar{A} \rightarrow \bar{B}$ is a naive fibration between naively fibrant objects and in W by the two out of six property, and so by hypothesis it is in \mathcal{B} . Therefore the diagonal filler exists. The converse follows from (a).

(c) This is straightforward from (b). \square

It is well known [2, Remark 1.4] that the local presentability assumption in Lemma 1.1 can be weakened, provided that the domains of the maps in \mathfrak{J} satisfy some property.

Remark 1.2. One can make variations in Lemma 1.1. For example, the path object argument devised by Quillen shows that the conclusion of (a) remains valid if instead of $\text{cell}(\mathfrak{J}) \subset W$ one requires that \mathcal{E} has a functorial naively fibrant replacement functor and every naively fibrant object has a naive path object. This new requirement also implies $\text{cell}(\mathfrak{J}) \subset W$.

The connection between Smith's theorem and Lemma 1.1 is the following

Proposition 1.3. *Let \mathcal{E} , W and I be as in Theorem 0.1, satisfying c0 and c1. Furthermore, suppose that W has the weak invertibility property. Let \mathfrak{J} be a set of maps of \mathcal{E} . If $\text{cell}(\mathfrak{J}) \subset \text{cof}(I) \cap W$ and a map between naively fibrant objects which is both a naive fibration and in W is in $\text{inj}(I)$, then c2 holds and an object of \mathcal{E} is fibrant in the resulting model structure if and only if it is naively fibrant. Moreover, the fibrations between fibrant objects are the naive fibrations.*

Proof. Apply Lemma 1.1 to the weak factorization system $(\text{cof}(I), \text{inj}(I))$. \square

Suppose that our category \mathcal{E} is a closed category; then one may ask whether the model structure on \mathcal{E} is compatible with the monoidal product, that is, whether \mathcal{E} is a monoidal model category. To ask for this compatibility makes sense in the absence of the model category structure. We let \otimes be the monoidal product of \mathcal{E} , and for two objects X, Y of \mathcal{E} , we write Y^X for their internal hom. In the language of Lemma 1.1 we have

Proposition 1.4. *Let \mathcal{E} be a locally presentable category, I and \mathfrak{J} two sets of maps of \mathcal{E} , and W a class of maps of \mathcal{E} having the two out of six property. Suppose that $\text{cell}(\mathfrak{J}) \subset \text{cof}(I) \cap W$ and a map between naively fibrant objects which is both a naive fibration and in W is in $\text{inj}(I)$. Then the following are equivalent:*

(a) *for any maps $A \rightarrow B$ and $K \rightarrow L$ of $\text{cof}(I)$, the canonical map*

$$A \otimes L \bigcup_{A \otimes K} B \otimes K \rightarrow B \otimes L$$

is in $\text{cof}(I)$, which is in W if either one of the given maps is in W ;

(b) *for any maps $A \rightarrow B$ and $K \rightarrow L$ of $\text{cof}(I)$, the canonical map*

$$A \otimes L \bigcup_{A \otimes K} B \otimes K \rightarrow B \otimes L$$

is in $\text{cof}(I)$, and for every element $A \rightarrow B$ of I and every naive fibration $X \rightarrow Y$ between naively fibrant objects, the canonical map

$$X^B \rightarrow Y^B \times_{Y^A} X^A$$

is a naive fibration between naively fibrant objects.

If the domains of the elements of I are in $\text{cof}(I)$, then (b) can be replaced by

(b') *for any maps $A \rightarrow B$ and $K \rightarrow L$ of $\text{cof}(I)$, the canonical map*

$$A \otimes L \bigcup_{A \otimes K} B \otimes K \rightarrow B \otimes L$$

is in $\text{cof}(I)$, and for every element $A \rightarrow B$ of I and every naive fibration $X \rightarrow Y$ between naively fibrant objects, the canonical maps $X^A \rightarrow Y^A$ and $X^B \rightarrow Y^B$ are naive fibrations between naively fibrant objects.

Proof. (a) \Rightarrow (b) follows from Lemma 1.1(b). For (b) \Rightarrow (a), we may assume, by Lemma 1.1(b), that $A \rightarrow B$ is an element of I and $K \rightarrow L$ is in $\text{cof}(I) \cap W$. But then the claim follows from Lemma 1.1(b). (a) \Rightarrow (b') is straightforward and for (b') \Rightarrow (a) one uses Lemma 1.1(b) and the two out of six property of W . \square

2. APPLICATION: CATEGORIES ENRICHED OVER MONOIDAL SIMPLICIAL MODEL CATEGORIES

We denote by \mathbf{S} the category of simplicial sets, regarded as having the Quillen model structure. We let \mathbf{Cat} be the category of small categories. We say that an arrow $f : C \rightarrow D$ of \mathbf{Cat} is an **isofibration** if for any $x \in \text{Ob}(C)$ and any isomorphism $v : y' \rightarrow f(x)$ in D , there exists an isomorphism $u : x' \rightarrow x$ in C such that $f(u) = v$. The class of isofibrations is invariant under isomorphisms.

2.1. Monoidal simplicial model categories. The next result is [1, Lemma 1.31], except that we were not able to verify in the proof of *loc. cit.* "... that the objects ... corepresent the same functor", so we have changed the statement accordingly. Another reason for this change is that in *loc. cit.* there is no relation between the monoidal product of \mathbf{M} and the action of \mathbf{S} on \mathbf{M} , and we feel that these two structures have to be compatible.

Lemma 2.1. *Let \mathbf{M} be a monoidal model category with cofibrant unit. To give a simplicial model category structure on \mathbf{M} , with tensor, hom and cotensor which we denote by*

$$- * - : \mathbf{S} \times \mathbf{M} \rightarrow \mathbf{M}$$

$$\text{Map}(-, -) : \mathbf{M}^{op} \times \mathbf{M} \rightarrow \mathbf{S}$$

$$(-)^{(-)} : \mathbf{S}^{op} \times \mathbf{M} \rightarrow \mathbf{M}$$

*such that $- * -$ is a strong monoidal functor is to give a Quillen pair $F : \mathbf{S} \rightleftarrows \mathbf{M} : G$ such that F is strong monoidal.*

A monoidal model category with cofibrant unit satisfying the equivalent conditions of Lemma 2.1 will be referred to as a **monoidal \mathbf{S} -model category** [10, 4.2.20].

2.2. Classes of \mathbf{M} -functors and the main result. Let \mathbf{M} be a monoidal model category with cofibrant unit I . We denote by \mathbf{MCat} the category of small \mathbf{M} -categories. If S is a set, we denote by $\mathbf{MCat}(S)$ (resp. $\mathbf{MGraph}(S)$) the category of small \mathbf{M} -categories (resp. \mathbf{M} -graphs) with fixed set of objects S . There is a free-forgetful adjunction

$$F_S : \mathbf{MGraph}(S) \rightleftarrows \mathbf{MCat}(S) : U_S$$

We denote by ε^S the counit of this adjunction. Every function $f : S \rightarrow T$ induces an adjoint pair $f_! : \mathbf{MCat}(S) \rightleftarrows \mathbf{MCat}(T) : f^*$. If \mathcal{K} is a class of maps of \mathcal{V} , we say that a \mathcal{V} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is **locally in \mathcal{K}** if for each pair $x, y \in \mathcal{A}$ of objects, the map $f_{x,y} : \mathcal{A}(x, y) \rightarrow \mathcal{B}(f(x), f(y))$ is in \mathcal{K} .

We have a functor $[-]_{\mathbf{M}} : \mathbf{MCat} \rightarrow \mathbf{Cat}$ obtained by change of base along the symmetric monoidal composite functor

$$\mathbf{M} \longrightarrow Ho(\mathbf{M}) \xrightarrow{Ho(\mathbf{M})(I, -)} Set$$

Definition 2.2. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism in \mathbf{MCat} .

1. The morphism f is a **DK-equivalence** if f is locally a weak equivalence of \mathbf{M} and $[f]_{\mathbf{M}} : [\mathcal{A}]_{\mathbf{M}} \rightarrow [\mathcal{B}]_{\mathbf{M}}$ is essentially surjective.
2. The morphism f is a **DK-fibration** if f is locally a fibration of \mathbf{M} and $[f]_{\mathbf{M}}$ is an isofibration.
3. The morphism f is called a **trivial fibration** if it is a DK-equivalence and a DK-fibration.
4. The morphism f is called a **cofibration** if it has the left lifting property with respect to the trivial fibrations.

It follows from the definitions that (a) a map f is a DK-equivalence if and only if $Ho(f)$ is an equivalence of $Ho(\mathbf{M})$ -categories, and (b) an \mathbf{M} -functor is a trivial fibration if and only if it is surjective on objects and locally a trivial fibration of \mathbf{M} .

We denote by \mathcal{I} the \mathbf{M} -category with a single object $*$ and $\mathcal{I}(*, *) = I$. For an object X of \mathbf{M} , we denote by 2_X the \mathbf{M} -category with two objects 0 and 1 , and with $2_X(0, 0) = 2_X(1, 1) = I$, $2_X(0, 1) = X$ and $2_X(1, 0) = \emptyset$. When \mathbf{M} is cofibrantly generated, an \mathbf{M} -functor is a trivial fibration if and only if it has the right lifting property with respect to the saturated class generated by $\{\emptyset \rightarrow \mathcal{I}\} \cup \{2_X \xrightarrow{2_i} 2_Y, i \text{ generating cofibration of } \mathbf{M}\}$. We have the following fundamental result of J. Bergner.

Theorem 2.3. [3] *The category \mathbf{SCat} of simplicial categories admits a cofibrantly generated model structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations. A generating set of trivial cofibrations consists of*

(B1) $\{2_X \xrightarrow{2_j} 2_Y\}$, where j is a horn inclusion, and

(B2) inclusions $\mathcal{I} \xrightarrow{\delta_y} \mathcal{H}$, where $\{\mathcal{H}\}$ is a set of representatives for the isomorphism classes of simplicial categories on two objects which have countably many simplices in each function complex. Furthermore, each such \mathcal{H} is required to be cofibrant and weakly contractible in $\mathbf{SCat}(\{x, y\})$. Here $\{x, y\}$ is the set with elements x and y and δ_y omits y .

Recall from [14, Definition 3.3] the monoid axiom. Our main result is

Theorem 2.4. *Let \mathbf{M} be a combinatorial monoidal \mathbf{S} -model category having cofibrant unit and which satisfies the monoid axiom. Then \mathbf{MCat} admits a combinatorial model category structure in which the weak equivalences are the DK-equivalences, the cofibrations are the elements of $\text{cof}(\{\emptyset \rightarrow \mathcal{I}\} \cup \{2_X \xrightarrow{2_i} 2_Y, i \text{ generating cofibration of } \mathbf{M}\})$, the fibrant objects are the locally fibrant \mathbf{M} -categories and the fibrations between fibrant objects are the DK-fibrations. If the model structure on \mathbf{M} is right proper, then so is the one on \mathbf{MCat} .*

Proof. We shall apply Theorem 0.1 via Proposition 1.3. We take \mathcal{E} to be \mathbf{MCat} , \mathcal{W} to be the class of DK-equivalences and \mathcal{I} to be the set $\{\emptyset \rightarrow \mathcal{I}\} \cup \{2_X \xrightarrow{2_i} 2_Y, i \text{ generating cofibration of } \mathbf{M}\}$. Let $F : \mathbf{S} \rightleftarrows \mathbf{M} : G$ be the Quillen pair guaranteed by Lemma 2.1. (F, G) induces adjoint pairs $F' : \mathbf{SCat} \rightleftarrows \mathbf{MCat} : G'$ and $F' : \mathbf{SCat}(S) \rightleftarrows \mathbf{MCat}(S) : G'$, for every set S . The first G' functor preserves trivial fibrations, and the latter adjoint pair is a Quillen pair. We take \mathcal{J} to be the set $F'(B2) \cup \{2_X \xrightarrow{2_i} 2_Y, i \text{ generating trivial cofibration of } \mathbf{M}\}$. Then one readily checks that an \mathbf{M} -category is naively fibrant if and only if it is locally fibrant. We claim that an \mathbf{M} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ between locally fibrant \mathbf{M} -categories is a naive fibration if and only if it is a DK-fibration. For, let \mathbf{MCat}_f be the full subcategory of \mathbf{MCat} consisting of the locally fibrant \mathbf{M} -categories. By [9, 8.5.16] we have a natural isomorphism of functors

$$\eta : [-]_{\mathbf{S}} G' \cong [-]_{\mathbf{M}} : \mathbf{MCat}_f \rightarrow \mathbf{Cat}$$

such that for all $\mathcal{A} \in \mathbf{MCat}_f$, $\eta_{\mathcal{A}}$ is the identity on objects. This implies the claim.

Let now $j : X \rightarrow Y$ be a trivial cofibration of \mathbf{M} . We show that for every \mathbf{M} -category \mathcal{A} , in the pushout diagram

$$\begin{array}{ccc} 2_X & \xrightarrow{2_j} & 2_Y \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array}$$

the map $\mathcal{A} \rightarrow \mathcal{B}$ is a DK-equivalence. Let $S = Ob(\mathcal{A})$. This pushout can be calculated as the pushout

$$\begin{array}{ccc} F_S U_S \mathcal{A} & \longrightarrow & F_S \mathcal{X} \\ \downarrow \varepsilon_{\mathcal{A}}^S & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array}$$

where $U_S \mathcal{A} \rightarrow \mathcal{X}$ is a certain map of \mathbf{M} -graphs with fixed set of objects S . But then the map $\mathcal{A} \rightarrow \mathcal{B}$ is known to be locally a weak equivalence of \mathbf{M} , see [15, Proof of Proposition 6.3(1)]. Finally, it remains to prove that if $\delta_y : \mathcal{I} \rightarrow \mathcal{H}$ is a map belonging to the set B2 from Theorem 2.3 and \mathcal{A} is any \mathbf{M} -category, then in the pushout diagram

$$\begin{array}{ccc} F' \mathcal{I} & \xrightarrow{x} & \mathcal{A} \\ F' \delta_y \downarrow & & \downarrow \\ F' \mathcal{H} & \longrightarrow & \mathcal{B} \end{array}$$

the map $\mathcal{A} \rightarrow \mathcal{B}$ is a DK-equivalence. We factor the map δ_y as $\mathcal{I} \xrightarrow{\delta'_y} \mathcal{H}' \rightarrow \mathcal{H}$, where the simplicial category \mathcal{H}' has $\{x\}$ as set of objects and $\mathcal{H}'(x, x) = \mathcal{H}(x, x)$, and then we take consecutive pushouts:

$$\begin{array}{ccc} F' \mathcal{I} & \xrightarrow{x} & \mathcal{A} \\ F' \delta'_y \downarrow & & \downarrow j \\ F' \mathcal{H}' & \longrightarrow & \mathcal{A}' \\ \downarrow & & \downarrow \\ F' \mathcal{H} & \longrightarrow & \mathcal{B} \end{array}$$

By Lemma 2.5 the map δ'_y is a trivial cofibration in the category of simplicial monoids, and we claim that the map j is a trivial cofibration in $\mathbf{MCat}(Ob(\mathcal{A}))$. For, j is a trivial cofibration if and only if it has the left lifting property with respect to the class of fibrations. So let $\mathcal{C} \rightarrow \mathcal{D}$ be a fibration in $\mathbf{MCat}(Ob(\mathcal{A}))$. To give a diagonal filler in a diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{C} \\ j \downarrow & & \downarrow \\ \mathcal{A}' & \longrightarrow & \mathcal{D} \end{array}$$

is to give a diagonal filler in the composed diagram

$$\begin{array}{ccc} F' \mathcal{I} & \longrightarrow & \mathcal{C} \\ F' \delta'_y \downarrow & & \downarrow \\ F' \mathcal{H}' & \longrightarrow & \mathcal{D} \end{array}$$

The adjoint transpose of the latter diagram factors as

$$\begin{array}{ccccc} \mathcal{I} & \longrightarrow & \bullet & \longrightarrow & G' \mathcal{C} \\ \delta'_y \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}' & \longrightarrow & \bullet & \longrightarrow & G' \mathcal{D} \end{array}$$

where the middle vertical map is a fibration of simplicial monoids. The claim follows.

By Proposition 3.1 the map $\mathcal{A}' \rightarrow \mathcal{B}$ is a full and faithful inclusion, therefore the map $\mathcal{A} \rightarrow \mathcal{B}$ is locally a weak equivalence of \mathbf{M} . Applying the functor $[-]_{\mathbf{M}}$ to the diagram

$$\begin{array}{ccc} F'\mathcal{I} & \xrightarrow{x} & \mathcal{A} \\ F'\delta_y \downarrow & & \downarrow \\ F'\mathcal{H} & \longrightarrow & \mathcal{B} \end{array}$$

and taking into account that F' preserves DK-equivalences and $Ob(\mathcal{B}) = Ob(\mathcal{A}) \cup \{*\}$, it follows that $\mathcal{A} \rightarrow \mathcal{B}$ is a DK-equivalence as well.

Suppose now that \mathbf{M} is right proper. Using the explicit construction of pullbacks in \mathbf{MCat} , the description of the fibrations between fibrant objects and [4, Lemma 9.4], we conclude that \mathbf{MCat} is right proper. \square

Lemma 2.5. *Let \mathcal{A} be a cofibrant simplicial category. Then for each $a \in Ob(\mathcal{A})$ the simplicial monoid $a^*\mathcal{A} = \mathcal{A}(a, a)$ is cofibrant.*

Proof. Let $S = Ob(\mathcal{A})$. \mathcal{A} is cofibrant in \mathbf{SCat} if and only if it is cofibrant as an object of $\mathbf{SCat}(S)$. The cofibrant objects of $\mathbf{SCat}(S)$ are characterized in [7, 7.6]: they are the retracts of free simplicial categories. Therefore it suffices to prove that if \mathcal{A} is a free simplicial category then $a^*\mathcal{A}$ is a free simplicial category for all $a \in S$. There is a full and faithful functor $\varphi : \mathbf{SCat} \rightarrow \mathbf{Cat}^{\Delta^{op}}$ given by $Ob(\varphi(\mathcal{A})_n) = Ob(\mathcal{A})$ for all $n \geq 0$ and $\varphi(\mathcal{A})_n(a, a') = \mathcal{A}(a, a')_n$. Recall [7, 4.5] that \mathcal{A} is a free simplicial category if and only if (i) for all $n \geq 0$ the category $\varphi(\mathcal{A})_n$ is a free category on a graph G_n , and (ii) for all epimorphisms $\alpha : [m] \rightarrow [n]$ of Δ , $\alpha^* : \varphi(\mathcal{A})_n \rightarrow \varphi(\mathcal{A})_m$ maps G_n into G_m .

Let $a \in S$. The category $\varphi(a^*\mathcal{A})_n$ is a full subcategory of $\varphi(\mathcal{A})_n$ with object set $\{a\}$, hence it is free as well. A set $G_n^{a^*\mathcal{A}}$ of generators can be described as follows. An element of $G_n^{a^*\mathcal{A}}$ is a path from a to a in $\varphi(\mathcal{A})_n$ such that every arrow in the path belongs to G_n and there is at most one arrow in the path with source and target a . Since every epimorphism $\alpha : [m] \rightarrow [n]$ of Δ has a section, α^* maps $G_n^{a^*\mathcal{A}}$ into $G_m^{a^*\mathcal{A}}$. \square

Remark 2.6. One can change the assumptions of Theorem 2.4 and the general principle used in its proof, and obtain a similar outcome. For example, let \mathbf{M} be a cofibrantly generated monoidal \mathbf{S} -model category having cofibrant unit and which satisfies the monoid axiom. Suppose furthermore that

- (a) a transfinite composition of weak equivalences of \mathbf{M} is a weak equivalence,
- (b) \mathbf{M} satisfies the technical condition of [11, Theorem 2.1], and
- (c) in the Quillen pair $F : \mathbf{S} \rightleftarrows \mathbf{M} : G$ guaranteed by Lemma 2.1, the functor G preserves weak equivalences.

Then [9, 11.3.1] can be used to show that \mathbf{MCat} admits a cofibrantly generated model category structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations. Condition (b) can be relaxed, it was stated in this form in order to include examples such as compactly generated spaces, cf. [11].

3. PUSHOUTS ALONG FULLY FAITHFUL FUNCTORS

Let $(\mathcal{V}, \otimes, I)$ be a cocomplete closed category. We denote by $\mathcal{V}\mathbf{Cat}$ the category of small \mathcal{V} -categories and by $\mathcal{V}\mathbf{Graph}$ that of small \mathcal{V} -graphs. A \mathcal{V} -functor, or a map of \mathcal{V} -graphs, which is locally an isomorphism is said to be **full and faithful**. If S is a set, we denote by $\mathcal{V}\mathbf{Cat}(S)$ (resp. $\mathcal{V}\mathbf{Graph}(S)$) the category of small \mathcal{V} -categories (resp. \mathcal{V} -graphs) with fixed set of objects S . The category $\mathcal{V}\mathbf{Graph}(S)$ is a monoidal category with monoidal product \square_S and unit \mathcal{I}_S .

Proposition 3.1. [8, Proposition 5.2] *Let \mathcal{A} , \mathcal{B} and \mathcal{C} be three small \mathcal{V} -categories and let $i : \mathcal{A} \hookrightarrow \mathcal{B}$ be a full and faithful inclusion. Then in the pushout diagram of \mathcal{V} -categories*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & \mathcal{B} \\ f \downarrow & & \downarrow g \\ \mathcal{C} & \xrightarrow{i'} & \mathcal{D} \end{array}$$

the map $i' : \mathcal{C} \rightarrow \mathcal{D}$ is a full and faithful inclusion.

Proof. We shall construct \mathcal{D} explicitly, as was done in the proof of [8, Proposition 5.2]. On objects we put $Ob(\mathcal{D}) = Ob(\mathcal{C}) \sqcup (Ob(\mathcal{B}) - Ob(\mathcal{A}))$ and $\mathcal{D}(p, q) = \mathcal{C}(p, q)$ if $p, q \in Ob(\mathcal{C})$. For $p \in Ob(\mathcal{C})$ and $q \in (Ob(\mathcal{B}) - Ob(\mathcal{A}))$ we define

$$\mathcal{D}(p, q) = \int^{x \in Ob(\mathcal{A})} \mathcal{B}(x, q) \otimes \mathcal{C}(p, f(x))$$

For $p \in (Ob(\mathcal{B}) - Ob(\mathcal{A}))$ and $q \in Ob(\mathcal{C})$ we define

$$\mathcal{D}(p, q) = \int^{x \in Ob(\mathcal{A})} \mathcal{C}(f(x), q) \otimes \mathcal{B}(p, x)$$

For $p, q \in (Ob(\mathcal{B}) - Ob(\mathcal{A}))$ we define $\mathcal{D}(p, q)$ to be the pushout

$$\begin{array}{ccc} \int^{x \in Ob(\mathcal{A})} \mathcal{B}(x, q) \otimes \mathcal{B}(p, x) & \longrightarrow & \int^{x \in Ob(\mathcal{A})} \int^{y \in Ob(\mathcal{A})} \mathcal{B}(x, q) \otimes \mathcal{C}(f(y), f(x)) \otimes \mathcal{B}(p, y) \\ \downarrow & & \downarrow \\ \mathcal{B}(p, q) & \longrightarrow & \mathcal{D}(p, q) \end{array}$$

We shall describe a way to see that, with the above definition, \mathcal{D} is indeed a \mathcal{V} -category.

Let $(\mathcal{B} - \mathcal{A})^+$ be the preorder with objects all finite subsets $S \subset Ob(\mathcal{B}) - Ob(\mathcal{A})$, ordered by inclusion. For $S \in (\mathcal{B} - \mathcal{A})^+$, let \mathcal{A}_S be the full sub- \mathcal{V} -category of \mathcal{B} with objects $Ob(\mathcal{A}) \cup S$. Then $\mathcal{B} = \lim_{(\mathcal{B} - \mathcal{A})^+} \mathcal{A}_S$. On the other hand, a filtered colimit of full and faithful inclusions of \mathcal{V} -categories is a full and faithful inclusion. This is because the forgetful functor from $\mathcal{V}\mathbf{Cat}$ to $\mathcal{V}\mathbf{Graph}$ preserves filtered colimits [13, Corollary 3.4] and a filtered colimit of full and faithful inclusions of \mathcal{V} -graphs is a full and faithful inclusion. Therefore one can assume from the beginning that $Ob(\mathcal{B}) = Ob(\mathcal{A}) \cup \{q\}$, where $q \notin Ob(\mathcal{A})$.

Case 1: f is full and faithful. In this case the pushout giving $\mathcal{D}(q, q)$ is simply $\mathcal{B}(q, q)$, all the other formulas remain unchanged. Then to show that \mathcal{D} is a \mathcal{V} -category is straightforward.

Case 2: f is the identity on objects. The map i induces an adjoint pair

$$i_! : \mathcal{V}\mathbf{Cat}(Ob(\mathcal{A})) \rightleftarrows \mathcal{V}\mathbf{Cat}(Ob(\mathcal{B})) : i^*$$

One has

$$i_! \mathcal{A}(a, a') = \begin{cases} \mathcal{A}(a, a'), & \text{if } a, a' \in Ob(\mathcal{A}), \\ \emptyset, & \text{otherwise,} \\ I, & \text{if } a = a' = q, \end{cases}$$

and i factors as $\mathcal{A} \rightarrow i_! \mathcal{A} \rightarrow \mathcal{B}$, where $i_! \mathcal{A} \rightarrow \mathcal{B}$ is the obvious map in $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{B}))$. Then the original pushout can be computed using the pushout diagram

$$\begin{array}{ccc} i_! \mathcal{A} & \longrightarrow & \mathcal{B} \\ i_! f \downarrow & & \downarrow \\ i_! \mathcal{C} & \longrightarrow & \mathcal{D} \end{array}$$

in $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{B}))$. Next, we claim that \mathcal{D} can be calculated as the pushout, in the category ${}_{\mathcal{B}}\mathbf{Mod}_{\mathcal{B}}$ of $(\mathcal{B}, \mathcal{B})$ -bimodules in

$$(\mathcal{V}\mathbf{Graph}(Ob(\mathcal{B})), \square_{Ob(\mathcal{B})}, \mathcal{I}_{Ob(\mathcal{B})})$$

of the diagram

$$\begin{array}{ccc} \mathcal{B} \square_{i_! \mathcal{A}} \mathcal{B} & \xrightarrow{\mathcal{B} \square_{i_! \mathcal{A}} i_! f \square_{i_! \mathcal{A}} \mathcal{B}} & \mathcal{B} \square_{i_! \mathcal{A}} i_! \mathcal{C} \square_{i_! \mathcal{A}} \mathcal{B} \\ \downarrow & & \downarrow m \\ \mathcal{B} & \longrightarrow & \mathcal{D} \end{array}$$

For this we have to show that \mathcal{D} is a monoid in ${}_{\mathcal{B}}\mathbf{Mod}_{\mathcal{B}}$. We first show that $\mathcal{B} \square_{i_! \mathcal{A}} i_! \mathcal{C} \square_{i_! \mathcal{A}} \mathcal{B}$ is a monoid in ${}_{\mathcal{B}}\mathbf{Mod}_{\mathcal{B}}$. There is a canonical isomorphism

$$i_! \mathcal{C} \square_{i_! \mathcal{A}} i_! \mathcal{C} \cong i_! \mathcal{C} \square_{i_! \mathcal{A}} \mathcal{B} \square_{i_! \mathcal{A}} i_! \mathcal{C}$$

of $(i_! \mathcal{A}, i_! \mathcal{A})$ -bimodules which is best seen pointwise, using coends. This provides a multiplication for $\mathcal{B} \square_{i_! \mathcal{A}} i_! \mathcal{C} \square_{i_! \mathcal{A}} \mathcal{B}$ which is again best seen to be associative by working pointwise, using coends. To define a multiplication for \mathcal{D}

consider the cube diagrams

$$\begin{array}{ccccc}
 \mathcal{B} \cdot i_! \mathcal{A} \cdot \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{A} \cdot \mathcal{B} & \xrightarrow{\quad} & \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{A} \cdot \mathcal{B} & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{A} \cdot \mathcal{B} & \xrightarrow{\quad} & \mathcal{D} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{A} \cdot \mathcal{B} & \\
 \mathcal{B} \cdot i_! \mathcal{A} \cdot \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} & \xrightarrow{\quad} & \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} & & \\
 \searrow & \downarrow & \searrow & \downarrow & \\
 & \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} & \xrightarrow{\quad} & \mathcal{D} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} &
 \end{array}$$

and

$$\begin{array}{ccccc}
 \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{A} \cdot \mathcal{B} & \xrightarrow{\quad} & \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & \mathcal{D} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{A} \cdot \mathcal{B} & \xrightarrow{\quad} & \mathcal{D} \cdot_{\mathcal{B}} \mathcal{B} & \\
 \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} & \xrightarrow{\quad} & \mathcal{B} \cdot_{\mathcal{B}} \mathcal{D} & & \\
 \searrow & \downarrow & \searrow & \downarrow & \\
 & \mathcal{D} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} & \xrightarrow{\quad} & \mathcal{D} \cdot_{\mathcal{B}} \mathcal{D} &
 \end{array}$$

For space considerations we have suppressed tensors (always over $i_! \mathcal{A}$, unless explicitly indicated) from notation. The right face of the first cube is the same as the left face of the latter cube. Let PO_1 (resp. PO_2) be the pushout of the left (resp. right) face of the first cube diagram. Let PO_3 be the pushout of the right face of the second cube diagram. We have pushout digrams

$$\begin{array}{ccccc}
 PO_1 & \xrightarrow{\quad} & PO_2 & \xrightarrow{\quad} & PO_3 \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} & \longrightarrow & \mathcal{D} \cdot_{\mathcal{B}} \mathcal{B} \cdot i_! \mathcal{C} \cdot \mathcal{B} & \longrightarrow & \mathcal{D} \cdot_{\mathcal{B}} \mathcal{D}
 \end{array}$$

Using these pushouts and the fact that $\mathcal{B} \square_{i_! \mathcal{A}} i_! \mathcal{C} \square_{i_! \mathcal{A}} \mathcal{B}$ is a monoid one can define in a canonical way a map $\mu : \mathcal{D} \cdot_{\mathcal{B}} \mathcal{D} \rightarrow \mathcal{D}$. We omit the long verification that μ gives \mathcal{D} the structure of a monoid. The map μ was constructed in such a way that m becomes a morphism of monoids. The fact that \mathcal{D} has the universal property of the pushout in the category $\mathcal{V}\mathbf{Cat}(\text{Ob}(\mathcal{B}))$ follows from its definition.

Case 3: f is arbitrary. Let $u = \text{Ob}(f)$. We factor f as $\mathcal{A} \xrightarrow{f^u} u^* \mathcal{C} \rightarrow \mathcal{C}$, where $\text{Ob}(u^* \mathcal{C}) = \text{Ob}(\mathcal{A})$, $u^* \mathcal{C}(a, a') = \mathcal{C}(fa, fa')$ and f^u is the obvious map, and take consecutive pushouts:

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{i} & \mathcal{B} \\
 f^u \downarrow & & \downarrow \\
 u^* \mathcal{C} & \longrightarrow & \mathcal{A}' \\
 \downarrow & & \downarrow \\
 \mathcal{C} & \longrightarrow & \mathcal{D}
 \end{array}$$

Now apply cases 2 and 1. □

4. APPLICATION: LEFT BOUSFIELD LOCALIZATIONS OF CATEGORIES OF MONOIDS

This section was motivated by the paragraph “As we mentioned above,...in general.” on page 111 of [12].

4.1. The problem. Let \mathbf{M} be a suitable monoidal model category, $L\mathbf{M}$ a left Bousfield localization of \mathbf{M} which is itself a monoidal model category and $Mon(\mathbf{M})$ the category of monoids in \mathbf{M} . The problem is to induce on $Mon(\mathbf{M})$ a model category structure somehow related to $L\mathbf{M}$. As pointed out in [12], such a model structure exists if, for example, (a) $L\mathbf{M}$ satisfies the monoid axiom or (b) $Mon(\mathbf{M})$ has a suitable left proper model category structure. In order for (a) to be fulfilled one needs to know the (generating) trivial cofibrations of $L\mathbf{M}$. However, it often happens that one does not have an explicit description of them. For (b), the category of monoids in a monoidal model category is rarely known to be left proper (it is left proper if the underlying model category has all objects cofibrant, for instance, which seems to us too restrictive to work with).

4.2. Our solution. We shall propose below a solution to the above problem. We shall avoid left properness using Theorem 0.1 via Proposition 1.3, and we shall reduce the verification of the monoid axiom for $L\mathbf{M}$ to a smaller - and hopefully more tractable in practice, set of maps. The model categorical framework will be the ‘combinatorial’ analogue of the one of [12, Section 8].

It will be clear that the method could potentially be applied to other structures than monoids.

4.2.1. Recollections on enriched left Bousfield localization. Let \mathcal{V} be a monoidal model category and \mathbf{M} a model \mathcal{V} -category with tensor, hom and cotensor denoted by

$$\begin{aligned} - * - &: \mathcal{V} \times \mathbf{M} \rightarrow \mathbf{M} \\ Map(-, -) &: \mathbf{M}^{op} \times \mathbf{M} \rightarrow \mathcal{V} \\ (-)^{(-)} &: \mathcal{V}^{op} \times \mathbf{M} \rightarrow \mathbf{M} \end{aligned}$$

Let S be a set of maps of \mathbf{M} between cofibrant objects.

Definition 4.1. A fibrant object W of \mathbf{M} is *S -local* if for every $f \in S$ the map $Map(f, W)$ is a weak equivalence of \mathcal{V} . A map f of \mathbf{M} is an *S -local equivalence* if for every S -local object W and for some (hence any) cofibrant approximation \tilde{f} to f , the map $Map(\tilde{f}, W)$ is a weak equivalence of \mathcal{V} .

In the previous definition, if the map $Map(\tilde{f}, W)$ is a weak equivalence of \mathcal{V} , then for any other cofibrant approximation \tilde{g} to f , the map $Map(\tilde{g}, W)$ is a weak equivalence of \mathcal{V} [9, 14.6.6(1)].

Theorem 4.2. [1] Let \mathcal{V} be a combinatorial monoidal model category, \mathbf{M} a left proper, combinatorial model \mathcal{V} -category and S a set of maps of \mathbf{M} between cofibrant objects. Suppose that \mathcal{V} has a set of generating cofibrations with cofibrant domains. Then the category \mathbf{M} admits a left proper, combinatorial model category structure, denoted by $L_S\mathbf{M}$, with the class of S -local equivalences as weak equivalences and the same cofibrations as the given ones. The fibrant objects of $L_S\mathbf{M}$ are the S -local objects. $L_S\mathbf{M}$ is a model \mathcal{V} -category.

Suppose, moreover, that \mathbf{M} is a monoidal model \mathcal{V} -category which has a set of generating cofibrations with cofibrant domains. Let us denote by \otimes the monoidal product on \mathbf{M} . If $X \otimes f$ is an S -local equivalence for every $f \in S$ and every X belonging to the domains and codomains of the generating cofibrations of \mathbf{M} , then $L_S\mathbf{M}$ is a monoidal model \mathcal{V} -category.

Theorem 4.2 has the following immediate consequence.

Corollary 4.3. [1] Let \mathcal{V} be a left proper, combinatorial monoidal model category having a set of generating cofibrations with cofibrant domains. Let S be a set of maps of \mathcal{V} between cofibrant objects. Then $L_S\mathcal{V}$ is a monoidal model category.

4.2.2. The S -extended monoid axiom. Let \mathcal{V} be a monoidal model category and \mathbf{M} a monoidal model \mathcal{V} -category with monoidal product \otimes and tensor, hom and cotensor denoted as in 4.2.1. If $i : K \rightarrow L$ is a map of \mathcal{V} and $f : A \rightarrow B$ a map of \mathbf{M} , we denote by $i *' f$ the canonical map

$$L * A \bigcup_{K * A} K * B \rightarrow L * B$$

Let S be a set of maps of \mathbf{M} between cofibrant objects. For every $f \in S$, let $f = v_f u_f$ be a factorization of f as a cofibration u_f followed by a weak equivalence v_f .

Definition 4.4. We say that \mathbf{M} satisfies the *S -extended monoid axiom* if, in the notation of [14, Section 3], every map in

$$(\{\text{trivial cofibrations}\} \cup (\{\text{cofibrations}\} *' u_f)_{f \in S}) \otimes \mathbf{M}\text{-cof}_{\text{reg}}$$

is an S -local equivalence.

As usual [14, Lemma 3.5(2)], if \mathcal{V} and \mathbf{M} are cofibrantly generated and every map in

$$(\{\text{generating trivial cofibrations}\} \cup (\{\text{generating cofibrations}\} *' u_f)_{f \in S}) \otimes \mathbf{M}\text{-cof}_{\text{reg}}$$

is an S -local equivalence, then the S -extended monoid axiom holds.

Let now $\text{Mon}(\mathbf{M})$ be the category of monoids in \mathbf{M} and

$$T : \mathbf{M} \rightleftarrows \text{Mon}(\mathbf{M}) : U$$

the free-forgetful adjunction.

Definition 4.5. A monoid M in \mathbf{M} is **TS -local** if $U(M)$ is S -local. A map f of monoids in \mathbf{M} is a **TS -local equivalence** if $U(f)$ is an S -local equivalence.

Theorem 4.6. Let \mathcal{V} be a combinatorial monoidal model category having a set of generating cofibrations with cofibrant domains. Let \mathbf{M} be a left proper, combinatorial monoidal model \mathcal{V} -category which has a set of generating cofibrations with cofibrant domains. Let S be a set of maps of \mathbf{M} between cofibrant objects. Let us denote by \otimes the monoidal product on \mathbf{M} . Suppose that $X \otimes f$ is an S -local equivalence for every $f \in S$ and every X belonging to the domains and codomains of the generating cofibrations of \mathbf{M} and that \mathbf{M} satisfies the S -extended monoid axiom.

Then the category $\text{Mon}(\mathbf{M})$ admits a combinatorial model category structure, denoted by $L_{TS}\text{Mon}(\mathbf{M})$, with TS -local equivalences as weak equivalences and with $T(\{\text{cofibrations}\})$ as cofibrations. The fibrant objects of $L_{TS}\text{Mon}(\mathbf{M})$ are the TS -local objects.

Proof. We shall apply Theorem 0.1 via Proposition 1.3. We take \mathcal{E} to be $\text{Mon}(\mathbf{M})$, \mathcal{W} to be the class of TS -local equivalences and \mathcal{I} to be the set $T(\{\text{generating cofibrations}\})$. Notice that a map g of monoids in \mathbf{M} belongs to $\text{inj}(T(\{\text{generating cofibrations}\}))$ if and only if $U(g)$ belongs to $\text{inj}(\{\text{generating cofibrations}\})$. We take \mathfrak{J} to be

$$T(\{\text{generating trivial cofibrations of } \mathbf{M}\} \cup \{\text{generating cofibrations} *' u_f\}_{f \in S})$$

The fact that $\text{cell}(\mathfrak{J}) \subset \text{cof}(\mathcal{I}) \cap \mathcal{W}$ is guaranteed by [14, Proof of Lemma 6.2] and hypothesis. One can check that a monoid in \mathbf{M} is naively fibrant if and only if it is TS -local. Let g be a map of monoids in \mathbf{M} between TS -local monoids, such that g is both a TS -local equivalence and a naive fibration. Then $U(g)$ is a trivial fibration. \square

Corollary 4.7. Let \mathcal{V} be a left proper, combinatorial monoidal model category having a set of generating cofibrations with cofibrant domains. Let S be a set of maps of \mathcal{V} between cofibrant objects. Suppose that \mathcal{V} satisfies the S -extended monoid axiom. Then the category of monoids in \mathcal{V} admits a combinatorial model category structure with TS -local equivalences as weak equivalences, $T(\{\text{cofibrations}\})$ as cofibrations and TS -local objects as fibrant objects.

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DEPARTMENT OF MATHEMATICS AND STATISTICS,
MASARYK UNIVERSITY, KOTLÁŘSKÁ 2,
602 00 BRNO, CZECH REPUBLIC
E-mail address: `stanculescu@math.muni.cz`